DOCUMENT RESUME

ED 315 297 SE 051 178

AUTHOR Cannon, Lawrence O.; Elich, Joe
TITLE Some Little Night Problems.
PUB DATE Oct 89
NOTE 9p.; Paper presented at the Annual Meeting of the American Mathematical Association of Two-Year Colleges (15th, Baltimore, MD, October 1989).
PUB TYPE Reports - Descriptive (141)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Algebra; *College Mathematics; Higher Education; Mathematical Concepts; *Mathematical Formulas; *Mathematical Logic; Mathematics Education; Mathematics Materials; *Mathematics Tests; *Problem Sets; Problem Solving

ABSTRACT

In most mathematics problem solving work, students' motivation comes from trying to please their teachers or to earn a good grade. The questions students must tackle are almost never generated by their own interest. Seven open-ended college algebra-level problems are presented in which the solution of one question suggests other related questions. Problems and successive questions are for: (1) Prime Numbers; (2) Pythagorean Triples; (3) Iteration; (4) Golden Ratio in Logarithms; (5) Equilateral Triangles; and (6) Mathematical Inductions. (YP)

SOME LITTLE NIGHT PROBLEMS

We have borrowed, unfairly perhaps, the title of this presentation from Mozart's *Eine Kleine Nachtmusik*. We intend no reflection on the beauty of Mozart's music, but we are more interested in the task of engaging students in mathematics to the point that Mozart's music (or any of the music that more frequently assaults the ears of our students) may be resented because it interferes with their concentration. We want to see our students intrigued by mathematics problems in a way that most textbook practice exercises do not demand, caught up by "little night problems" that can occupy the night hours as rewardingly as Mozart has for music lovers.

The great majority of exercises that appear in the standard textbooks are designed to help students to achieve skills-mastery. Most of the others are intended to help students develop "problem-solving skills," whatever that may mean to the textbook authors. These are laudable goals and truly essential for our students. It also seems likely that a great many of our students can anticipate at most a modicum of skills and, we would surely hope, some problem solving ability, despite our best efforts as teachers to share our collective wisdom.

At the same time, however, most of us are fortunate enough to teach at least a few students who are capable of much more than minimal skills acquisition. Such students are not well-served if we do nothing more than assign them lots of standard, or even "challenging," exercises. While many talented students thrive on a regimen of working through great quantities of exercises, and certainly succeed admirably in their school careers, they may never encounter any real mathematics or experience the excitement of the discipline that lured most of us into our profession.

Novelist Walker Percy in an essay, "The Loss of the Creature," decries the failure of our educational system to *engage* students, to allow them a genuine encounter with the objects of education. Compare two educational experiences of the sort suggested by Percy's essay:

(1) A frog, properly labeled as a specimen of a particular species, is laid out on a dissecting table, together with a scalpel and a list of mimeographed questions to be answered in order.

(2) A student, with the guidance of a teacher, finds and identifies her own frog and proceeds to try to determine why one particular valve opens in this direction rather than the other, expected, way.

Those of us who must deal with the realities of day-to-day teaching, with the number and diversity of student abilities and interests, recognize the necessity, the absolute unavoidability of providing experiences like (1). At the same time, we have no illusions about which kind of experience is more significant for the learner.

In mathematics, our laboratory is largely problem-solving, with pencil and paper augmented by calculators, and sometimes computers, as our dissecting tools. In our laboratories, we cannot avoid laying out frogs and listing questions to be answered, but when we identify a student with the curiosity and interest to benefit from an individual field trip, we should surely extend an invitation and provide guidance.

In almost all of their problem solving work, our students are trying to solve *our* problems, not theirs. Their motivation comes from trying to please their teachers or to earn a good grade. The questions students must tackle are almost never generated by their own interest. Mathematics is presented as a finished edifice, all done, built in the past by geniuses far removed from ordinary experience. We communicate no sense of the vitality and vigor of a living subject, growing faster today than ever before, responding to present needs and intellectual curiosity. But just as a knowledgable biology teacher can arouse curiosity about some structural anomaly, if we are alert to opportunities, we can sometimes engage students in explorations of their own. They can be led to ask, "What happens if...?" or "Is it always true that...?" or "Why can't...?," and the questions are truly their own, and the answers they find constitute their own mathematics.

Our proposal, tnough certainly not novel, is simply that we invite some of our better students to "ponder mathematics," by tackling questions that may not even have an "answer." The problems we consider here we have used both with individual students and with some classes. Some classes lend themselves well to collective pondering; other classes are too tightly scheduled to allow much exploration as a group.

Our goal is college algebra level problems that are open-ended, where the resolution of one question suggests other, related, questions. The real value of these problems is in thinking hard about what is happening, making guesses, validating or invalidating guesses , or even just looking at enough examples to strengthen the conviction that a guess *should* be true. We know that we have succeeded when we get a complaint that someone has stayed up beyond the normal hour, or even better, has awakened during the night to think about a problem. We present herewith a sampling of problems that have bothered us to the extent that they became our own Night Problems. We claim no more originality or ownership of these problems than that earned by affection and familiarity.

Night Problem #1. PRIMES IN UNEXPECTED PLACES

Consider the function $f(n) = \sqrt{24n+1}$, $n = 1, 2, 3, \ldots$. Some of the values in the range of the function are positive integers. For example,

$f(1) = 5$, $f(2) = 7$, $f(5) = 11$, $f(7) = 13$, $f(12) = 17$, $f(15) = 19$.

What kind of integers are included in the range of f?

One obvious observation is that all of the values listed above are odd, but 15 is missing. Assuming that we have tried 8, 9, 10, and 11 for n without getting 15, it should be clear that 15 isn't in the range, so the range cannot include all odd numbers greater than 3.

Does the range of f include any even integers? In particular, is there an integer n for which $f(n) = 20$? or an integer n for which $n = 44$? Is there an integer n for which $f(n)$ is ever even?

All of the integer range values listed above are prime:

5, 7, 11, 13, 17, 19.

Does this continue? Is there an integer n for which $n = 23$? 29? for any given odd prime?

Are primes the only integers in the range of f? Is there an integer n for which $n = 21$? 25? 27? 33? 35? 49?

How can we distinguish between the odd integers for which there is a solution, the integers that are in the range of f, and the odd integers that do not belong to the range of f?

Night Problem #2 PYTHAGOREAN TRIPLES (in or out of triangles)

As a way to get students into the problem, we usually begin with a particular family of solutions. Almost all students can come up with the examples of 3,4,5 and 5,12,13 triangles, in both of which the hypotenuse differs from the longer leg by 1. If we look at sides a,b,c and perimeter P and area A, then we ask students to fill in the following table, considering some questions:

Row	a	b	c	P	2A
1	3	4	5	12	12
2	5	12	13	30	60
3					
4	9	40	41	90	360
5					

If one divisor of P is a, what is the other divisor?
If one divisor of 2A is P, what is the other divisor?
What are the entries on Row 3? on Row 5? on Row n?
Do your formulas on Row n satisfy the equation $a^2+b^2=c^2$?

A much less well-known family comes from another table, which may be more challenging to decipher. We may ask, for example:

What are the factors of a? of b? of P? How are a and P related?
Is c always prime?
What is the difference between c and b?
What are the entries on Row 3? on Row 5? on Row n?
Do your formulas on Row n satisfy the equation $a^2+b^2=c^2$?

Row	a	b	c	P
1	15	8	17	40
2	21	20	29	70
3				
4	39	80	89	208
5				

The usual number-theoretic derivation for the standard generating formulas of primitive Pythagorean triples are not appropriate for most algebra classes. There is a simple geometric exercise, however, which leads students to their own derivation. Consider the unit circle, with equation $x^2+y^2=1$, and the line L_m through the point (0,-1) with slope m: y = mx-1. Then L_m intersects the circle in the point (0,-1) and another point, P(m). If the slope is any number greater than 1, then P(m) is in the first quadrant. See Figure 1. Then we ask:

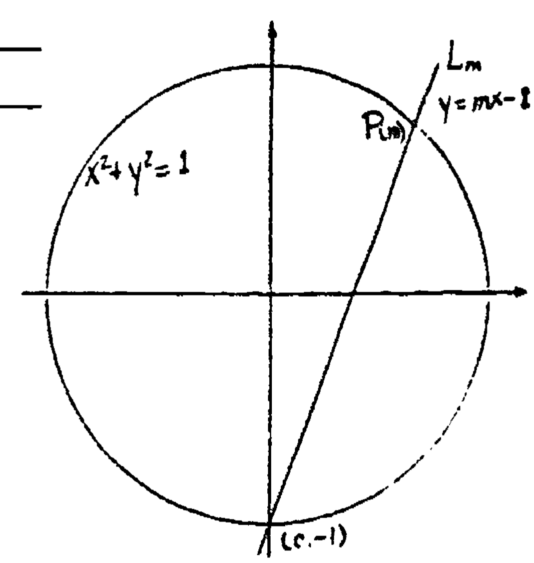

Figure 1

What are the coordinates of P(m) in terms of the slope m?
If m is a rational number, say m = u/v, must the coordinates of P(m) in terms of u and v be rational? have the same denominator? If P(m) is, say, (a/c, b/c), what can you say about a^2+b^2?

Now suppose that a first quadrant point P on the unit circle has rational coordinates, say P = (a/c, b/c). What is the slope of the line through P and the point (0,-1)?

From the answers to the previous two questions, can you find a formula that generates all Pythagorean triples?

Night Problem #3 EXXING IN AND EXXING OUT (Solutions by Iteration)

One of the most familiar calculator keys is the $\boxed{x^2}$ key. Beginning with any number between -1 and 1 and repeatedly pressing the $\boxed{x^2}$ key, we get a sequence $\{x_n\}$ that converges to zero, a process we sometimes describe in class as "exxing in to 0." Starting outside of the inteval [-1, 1] gives a sequence that simply "exxes out," diverges. Just being familiar with the behavior of these sequences can help a student later in understanding certainly limits, but more intriguing questions are available immediately.

With the calculatc. in radian mode, enter some number and press the $\boxed{\cos}$ key. Now continue to press the same key and watch the display. Pretty soon the display will show only numbers beginning 0.7, and after a few more steps, all displays begin 0.73, then 0.739, 0.73908, and finally the display doesn't change any more, constantly displaying a number, which to ten decimal places, is 0.73908133. Thus, to calculator accuracy, we have the following:

$$\cos 0.73908133 = 0.73908133.$$

In other words, we have exxed in on a calculator solution to the equation cos x = x.

Make a sketch showing the graphs of $y = x$ and $y = \cos x$. How does the intersection point of the two graphs relate to the number 0.73909133 *we found above?*

Now enter a number, change sign, and press the $\boxed{e^x}$ key; that is, evaluate the function $f(x) = e^{-x}$, and iterate.

Make a sketch showing the graphs of $y = x$ and $y = e^{-x}$. What are the coordinates of the intersection point of the two graphs?

Make a sketch showing the graphs of $y = x$ and $y = (1/e^x)^2 = e^{-x^2}$

Can you find the best calculator solution to the equation $e^{-x^2} = x$?

Make a sketch showing the graphs of $y = \sin x$ and $y = 1 - x$. Can the same iteration procedure be modified to solve the equation $\sin x = 1 - x$?

The only solution to the equation sin x = x is the number 0.

Make a sketch showing the graphs of $y = \sin x$ and $y = x$. What happens if we iterate the function $f(x) = \sin x$ (that is, repeatedly press the $\boxed{\sin}$ key)? WARNING: Be prepared to exercise extreme patience. Access to a computer and running a program with a loop to evaluate sin x many times will show that thousands of iterations makes very little progress toward the solution of the equation sin x = x. You may want to avoid just starting a loop with instructions to stop when the iterations get to within a certain distance of zero.

What is the difference between the graphs of functions $y = f(x)$ for which we can use iteration to efficiently solve the equation $f(x) = x$ (as we did for $\cos x$, e^{-x}, etc.) and the graph of $y = \sin x$?

Look at the graph of $y = \tan x$ and from the nature of the graph, guess whether iteration of the tangent function is going to find the solution to the equation $\tan x = x$.

What functions f generate sequences that converge (reasonably quickly) to a solution of $f(x) = x$?

Does it matter where we begin the sequence?

Night Problem #4 NIGHTMARES AND DESCENT INTO CHAOS (More on Iteration)

Iteration schemes can solve quite efficiently, transcendental equations for which we have no effective algebraic procedures, as with the equation $\cos x = x$. Students can also play with iterations to find solutions to equations that can be solved directly in exact form.

Sometimes we have to manipulate an equation to get a function that generates a convergent sequence. Consider the fairly simple equation $2^x = x^2$, for which two solutions, $x_1 = 2$ and $x_2 = 4$, are obvious. From the graph in Figure 2, we can see that there is also another solution x_3 which isn't easily found by inspection. Manipulating the equation $2^x = x^2$ to isolate x, we can write

$$x = \sqrt{2^x} = 2^{x/2} \quad \text{or} \quad x = \frac{2 \ln x}{\ln 2},$$

suggesting two different functions for iteration:

$$f_1(x) = \sqrt{2^x} = 2^{x/2} \quad \text{and} \quad f_2(x) = \frac{2 \ln x}{\ln 2}$$

It is easy to verify that iterating f_1 exxes in on the solution $x_1 = 2$ and iterating f_2 exxes in on the solution $x_2 = 4$. To find x_3 requires some subterfuge, suggested by the following :

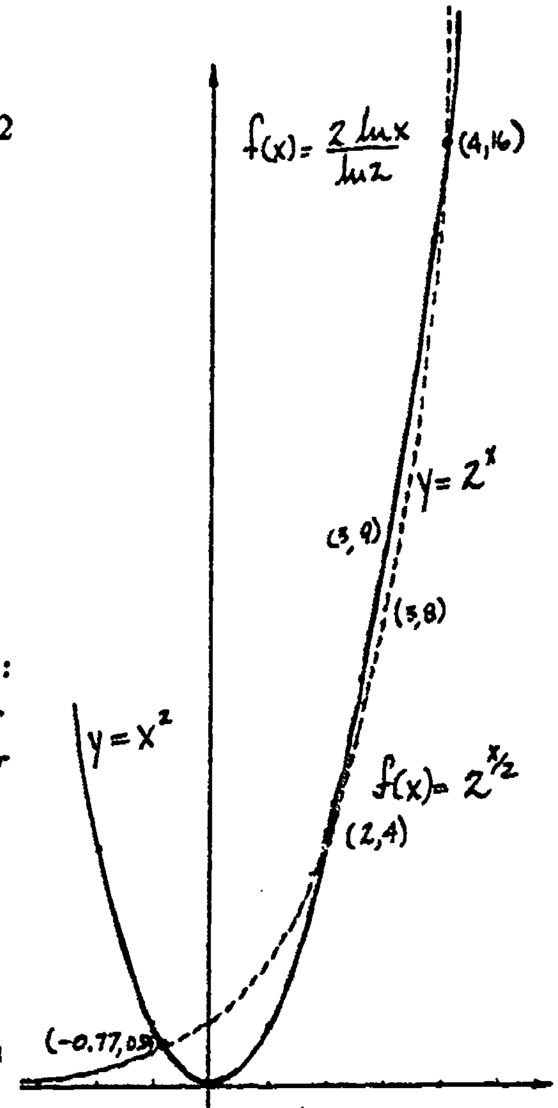

Figure 2

How is x_3 related to the intersections of the graphs of $y = x^2$ and $y = 2^{-x}$? What function might be iterated for exxing in on the pertinent solution to the equation $2^x = x^2$? What is x_3?

After looking at an equation for which we can see simple solutions, how about a slightly more complicated situation, say the transcendental equation, $2^x = x^{10}$? By comparing with $2^x = x^2$, it should be clear that there are two solutions between -1 and 1, and since the exponential function increases faster than any polynomial, there must be another solution for some larger positive x.

Can we iterate functions similar to f_1 and f_2 to find the positive solutions to $2^x = x^{10}$?

Could we use a graphing calculator or commercial graphing software (say, Toolkit, for example) to locate the solutions to $2^x = x^{10}$?

The answer to the last questions deserves some comment. Iterating the function $F(x) = 10(\ln x)/(\ln 2)$ quickly exxes in on the solution $x_3 = 58.77010594$. The calculator readily yields a corresponding y-coordinate of about 4.9×10^{17}, which our students recognize as "a pretty big number." Could we expect to see this intersection with Toolkit? Why not take a moment to discuss with students the kind of numbers we often toss around without so casually? In considering the graphs of $y = 2^x$ and $y = x^{10}$, just how large a picture would it take to show the third intersection? A little computation shows that if our graph has a y-axis scale of 10 units per inch, then the y-value of the third intersection point would be about 5000 *times the distance from the earth to the sun!*

In playing with equations to find iteration schemes for locating roots, we shouldn't be surprised to run into some equations for which nothing seems to work very well. Can students profitably explore some of the limits of iteration? We've already seen that straightforward iteration doesn't do much to locate the root of the equation $\sin x = x$.

To better see what works and what doesn't, we like to look at problems for which we already know the answers. Take, for example, the function $f(x) = \frac{3}{4}x^2 - 1$. Iterating f(x), we get a sequence similar to some we have already seen.

Starting with a number in the display such as 1.324 and iterating f(x), the numbers don't settle down readily; in fact, they seem to be bouncing back and forth between numbers beginning -0.8 and others beginning -0.4. We appear to have two subsequences, and furthermore, they seem to be moving closer together (we'd hope toward a common goal, converging to a solution to the equation $f(x) = x$). Unfortunately, progress is so slow that we can't even be confident that the subsequences are getting together eventually.

What about the possibility of doing some averaging? Store one of the display values, call it x_1, in memory, then get x_2 by iterating f (that is, $x_2 = f(x_1)$). Then take the average of x_1 and x_2, iterate, store, iterate, and average again, and see what happens.

The convergence is very rapid toward the best calculator approximation to -2/3, and it is easy to verify that $f(-2/3) = -2/3$. The solution we have obtained satisfies the equation, $\frac{3}{4}x^2 - 1 = x$. With a quadratic equation, the quadratic formula provides an exact solution; in this case, the roots are -2/3 and 2. While many initial values give a sequence converging (at least in a leisurely fashion) to -2/3, how can we get a sequence exxing in on 2? Finding values is surprisingly difficult. Starting with 2 or -2, we trivially get a constant sequence. Beginning with 2.0001, however, the sequence exxes out (diverges to ∞), and beginning with 1.9999 we get a sequence that exxes in on -2/3.

Changing the form of the function can change the nature of the solution. In this case, if we divide through by x, we get $\frac{3}{4}x - \frac{1}{x} = 1$, which can be rearranged into the form $x = \frac{4}{3}(1 + \frac{1}{x})$. Then using $F(x) = \frac{4}{3}(1 + \frac{1}{x})$ and iterating, most initial values generate sequences exxing in on 2 and it is as difficult to get a sequence converging to -2/3 as it was to get one converging to 2 for the function f.

Given a quadratic equation $ax^2 + bx + c = 0$, $b \neq 0$, suppose

$$f(x) = -(ax^2 + c)/b \quad \text{and} \quad F(x) = -(b + c/x)/a.$$

Will iteration of $f(x)$ converge to one of the roots of the quadratic equation, and if so, can you predict which one? Does the function $F(x)$ generate sequences converging to the other root?

The answer to this question is dramatically NO, as can be illustrated by iterating the simple function $f(x) = 2x^2 - 1$ (for which we would hope to get one of the roots of the equation $2x^2 - 1 = x$, namely 1 or -1/2). As expected, beginning with 0 or ± 1, we soon get the constant sequence 1, 1, 1, ..., and ± 0.5 leads to -0.5, -0.5, -0.5, ... Virtually any other initial value, however, generates a genuinely chaotic sequence in the sense that there is no convergence; the sequence wanders around through the interval (-1, 1), repeatedly returning to numbers *near* previous values, but never hitting them exactly. For example, beginning with $x_1 = 0.5100$, in twenty-seven steps we are within 0.0015 of x_1 (0.5015), then after another anothe- forty-three steps we are within 0.0013 of x_1 (0.4987), but no nearer a solution. To further aggravate the situation, the iteration is highly sensitive to the starting point, the initial value. With an initial value of $x_1 = 0.509$, in ten steps we have x_{10}

$= 0.9797$, but an initial value of $x_1 = 0.511$ leads in ten steps to $x_{10} = -0.8480$. That is, starting with two numbers within 0.002 of each other leads in ten iterations to numbers that differ by almost 2, growing apart by a *factor of a thousand.* Then in a few more steps the sequences almost match up again, before wandering off, each listening to the beat of a different drummer.

Considering the quadratic functions $f_1(x) = \frac{3}{4}x^2 - 1$ and $f_2(x) = 2x^2 - 1$, we have seen that iteration of f_1 exxes in on a solution of the equation $f_1(x) = x$, while iteration of f_2 leads to chaos.

For what coefficients a can we iterate the function $f(x) = ax^2 - 1$ and find a solution to the equation $f(x) = x$?

The list of Little Night Problems is, we hope, without end We are confident, at least, that the store of intriguing and fascinating questions far exceeds our capacity in our lifetime. But these are not mere puzzles for intellectual entertainment. If we can rework these problems into forms accessible to our students, then they can find their own problems, explore some of their capabilities and limitations, strengthen their mathematical and analytical muscles, and perhaps even create their own mathematics.

Students, like teachers, have their own particular interests and certainly function at various levels of mathematical sophistication. A problem that opens up a world of fascination for one student may have no real interest for a classmate who has essentially the same mathematical talent. The problems areas listed below call on different kinds of algebraic preparation and skills. Some require considerable patience and dedication while others may be cracked quite quickly (depending, of course, on who is doing the cracking). We invite you to sample and tailor them to the needs of your students.

Many of the night problems listed here are included in some form as "Explore and Discover" exercises in our precalculus text and are reprinted here by permission of Scott, Foresman and Company.

#5. From Logarithms to the Golden Ratio

If $\log_9 x = \log_{12} y = \log_{16}(x + y)$, find the value of the ratio y/x.

Consider the logarithmic equation $\log_a x = \log_b y = \log_c(x + y)$. Show that the ratio y/x has the same value for each of the following sequences a, b, c:

(*i*) 2, 6, 18 (*ii*) 4, 6, 9 (*iii*) 3, 12, 48.

What kind of sequence is a, b, c, in each of the above examples. Does every such sequence a, b, c give the same value for the ratio y/x from the same logarithmic equation, $\log_a x = \log_b y = \log_c(x + y)$?

#6. *Hinged Equilateral Triangles.*

Given the vertical distances $|\overline{AD}| = 7$ and $|\overline{CE}| = 11$ as in the figure, find an equilateral triangle that can be fit as shown (assuming the horizontal distance $|\overline{DE}|$ can be adjusted as needed).

What if $|\overline{AD}| = 9$ and $|\overline{CE}| = 12$? Show that there is still a solution. Are there any limitations on possible values for $|\overline{AD}|$ and $|\overline{CE}|$? Are there values for which the ratio of the horizontal distance to the length of the triangle is rational?

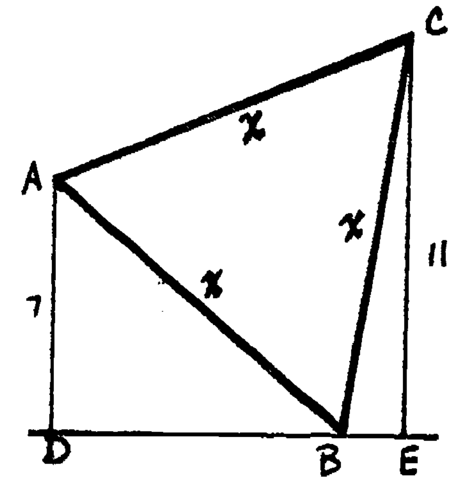

Figure 3

#7. Why Mess with Mathematical Induction?

Mathematical induction is almost always presented to students as a mysterious technique to prove that some formulas, usually about sums, are true, when it is obvious to those who look at a few examples, in the words of Li'l Abner, "...as any fool can plainly see."

If students are given a chance to make their own guesses about formulas, they may develop some appreciation, and a little more respect, for the discovery of valid formulas, especially when some of their "obvious" guesses turn out to be wrong.

7A. Given n points on a circle, connect each pair of points, and consider three functions:

C(n) is the <u>number of chords</u>.

D(n) is the <u>number of diagonals</u> of the polygon obtained connecting the points in order around the circle.

R(n) is the <u>number of regions</u> into which the chords di… the interior of the circle, assuming that no three chords have a common interior point.

Rough sketchs provide the following data:

n	1	2	3	4	5
$C(n)$	0	1	3	6	10
$D(n)$	0	0	0	2	5
$R(n)$	1	2	4	8	16

On the basis of the numbers in the table, guess a formula for R(n).

Look at the factors of the numbers for C(n) and guess a formula. It help to write the numbers as 2C(n)/2.

Extend the table to get enough data to make a guess for a formula for D(n). It may help to compare the numbers in the second row with the numbers in the first row (C(n)).

Are your formulas correct?

Proofs of the correctness of their guesses will certainly require thinking about induction in ways different than those found in most textbooks. Validating their guesses for C(n) and D(n) will be fairly straightforward for most curious students. Interestingly, however, unless we provided lots of help, *we have never had a student who made the correct guess for R(n).* The only reasonable guess on the basis of the given data requires doubling, which is simply not what is happening. Only after considering sums of binomial coefficients does the guess get better, and a proof remains a substantial challenge even after the correct guess.

7b. For a given set S of natural numbers and a given natural number k, define the <u>k-shift of S</u> denoted by S+k, to be the set obtained by adding k to every element of S: $S+k = \{ s+k \mid s \in S \}$, so if $S = \{3, 5, 9, 18\}$, then $S+4 = \{7, 9, 13, 22\}$.

Let $S_1 = \{1\}$ *and define* $S_{n+1} = [S_n+(n+1)] \cup \{2n+1\}$.

After writing out the first few sets, can you find a way to characterize the numbers that appear in a given S_n*? Hint: consider the numbers as sums involving n.*

Does every natural number appear in some (at least one) S_n*, or are there some natural numbers that don't ever show up in any* S_n*? Proof?*

A formal induction proof is probably too awkward for most students, but it's not unreasonable to give a clear argument as to why the conclusion is valid.